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MATHEMATICS
HIGHER LEVEL
PAPER 2

Candidate session number

Thursday 13 November 2014 (morning)

2 hours

Examination code

8	8	1	4	–	7	2	0	2

INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions in the boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the ***Mathematics HL and Further Mathematics HL*** *formula booklet* is required for this paper.
- The maximum mark for this examination paper is *[120 marks]*.

16 pages
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Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, for example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

SECTION A

*Answer **all** questions in the boxes provided. Working may be continued below the lines if necessary.*

1. *[Maximum mark: 6]*

Consider the two planes

$$\pi_1 : 4x + 2y - z = 8$$

$$\pi_2 : x + 3y + 3z = 3 .$$

Find the angle between π_1 and π_2, giving your answer correct to the nearest degree.

2. *[Maximum mark: 5]*

The wingspans of a certain species of bird can be modelled by a normal distribution with mean 60.2 cm and standard deviation 2.4 cm.

According to this model, 99 % of wingspans are greater than x cm.

(a) Find the value of x. *[2]*

In a field experiment, a research team studies a large sample of these birds. The wingspans of each bird are measured correct to the nearest 0.1 cm.

(b) Find the probability that a randomly selected bird has a wingspan measured as 60.2 cm. *[3]*

3. *[Maximum mark: 6]*

Consider the data set $\{2, x, y, 10, 17\}$, $x, y \in \mathbb{Z}^+$ and $x < y$.

The mean of the data set is 8 and its variance is 27.6.

Find the value of x and the value of y.

4. *[Maximum mark: 5]*

Two cyclists are at the same road intersection. One cyclist travels north at $20\,\text{km}\,\text{h}^{-1}$. The other cyclist travels west at $15\,\text{km}\,\text{h}^{-1}$.

Use calculus to show that the rate at which the distance between the two cyclists changes is independent of time.

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5. *[Maximum mark: 8]*

The lines l_1 and l_2 are defined as

$$l_1 : \frac{x-1}{3} = \frac{y-5}{2} = \frac{z-12}{-2}$$

$$l_2 : \frac{x-1}{8} = \frac{y-5}{11} = \frac{z-12}{6}.$$

The plane π contains both l_1 and l_2.

(a) Find the Cartesian equation of π. *[4]*

The line l_3 passing through the point $(4, 0, 8)$ is perpendicular to π.

(b) Find the coordinates of the point where l_3 meets π. *[4]*

6. *[Maximum mark: 6]*

Consider $p(x) = 3x^3 + ax + 5a$, $a \in \mathbb{R}$.

The polynomial $p(x)$ leaves a remainder of -7 when divided by $(x - a)$.

Show that only one value of a satisfies the above condition and state its value.

7. *[Maximum mark: 9]*

The seventh, third and first terms of an arithmetic sequence form the first three terms of a geometric sequence.

The arithmetic sequence has first term a and non-zero common difference d.

(a) Show that $d = \frac{a}{2}$. *[3]*

The seventh term of the arithmetic sequence is 3. The sum of the first n terms in the arithmetic sequence exceeds the sum of the first n terms in the geometric sequence by at least 200.

(b) Find the least value of n for which this occurs. *[6]*

8. *[Maximum mark: 7]*

A particle moves in a straight line such that its velocity, $v\,\text{m s}^{-1}$, at time t seconds, is given by

$$v(t) = \begin{cases} 5-(t-2)^2, & 0 \le t \le 4 \\ 3-\dfrac{t}{2}, & t > 4 \end{cases}.$$

(a) Find the value of t when the particle is instantaneously at rest. *[2]*

The particle returns to its initial position at $t = T$.

(b) Find the value of T. *[5]*

9. *[Maximum mark: 8]*

Compactness is a measure of how compact an enclosed region is.

The compactness, C, of an enclosed region can be defined by $C = \frac{4A}{\pi d^2}$, where A is the area of the region and d is the maximum distance between any two points in the region.

For a circular region, $C = 1$.

Consider a regular polygon of n sides constructed such that its vertices lie on the circumference of a circle of diameter x units.

(a) If $n > 2$ and even, show that $C = \frac{n}{2\pi}\sin\frac{2\pi}{n}$. *[3]*

If $n > 1$ and odd, it can be shown that $C = \frac{n\sin\frac{2\pi}{n}}{\pi\left(1+\cos\frac{\pi}{n}\right)}$.

(b) Find the regular polygon with the least number of sides for which the compactness is more than 0.99. *[4]*

(c) Comment briefly on whether C is a good measure of compactness. *[1]*

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(Question 9 continued)

*Do **NOT** write solutions on this page.*

SECTION B

*Answer **all** questions in the answer booklet provided. Please start each question on a new page.*

10. *[Maximum mark: 12]*

Consider the triangle PQR where $Q\hat{P}R = 30^\circ$, $PQ = (x+2)\,\text{cm}$ and $PR = (5-x)^2\,\text{cm}$, where $-2 < x < 5$.

(a) Show that the area, A cm^2, of the triangle is given by $A = \frac{1}{4}\left(x^3 - 8x^2 + 5x + 50\right)$. *[2]*

(b) (i) State $\frac{\mathrm{d}A}{\mathrm{d}x}$.

(ii) Verify that $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$ when $x = \frac{1}{3}$. *[3]*

(c) (i) Find $\frac{\mathrm{d}^2 A}{\mathrm{d}x^2}$ and hence justify that $x = \frac{1}{3}$ gives the maximum area of triangle PQR.

(ii) State the maximum area of triangle PQR.

(iii) Find QR when the area of triangle PQR is a maximum. *[7]*

*Do **NOT** write solutions on this page.*

11. *[Maximum mark: 10]*

The number of complaints per day received by customer service at a department store follows a Poisson distribution with a mean of 0.6.

(a) On a randomly chosen day, find the probability that

(i) there are no complaints;

(ii) there are at least three complaints. *[3]*

(b) In a randomly chosen five-day week, find the probability that there are no complaints. *[2]*

(c) On a randomly chosen day, find the most likely number of complaints received. Justify your answer. *[3]*

The department store introduces a new policy to improve customer service. The number of complaints received per day now follows a Poisson distribution with mean λ.

On a randomly chosen day, the probability that there are no complaints is now 0.8.

(d) Find the value of λ. *[2]*

12. *[Maximum mark: 11]*

Ava and Barry play a game with a bag containing one green marble and two red marbles. Each player in turn randomly selects a marble from the bag, notes its colour and replaces it. Ava wins the game if she selects a green marble. Barry wins the game if he selects a red marble. Ava starts the game.

Find the probability that

(a) Ava wins on her first turn; *[1]*

(b) Barry wins on his first turn; *[2]*

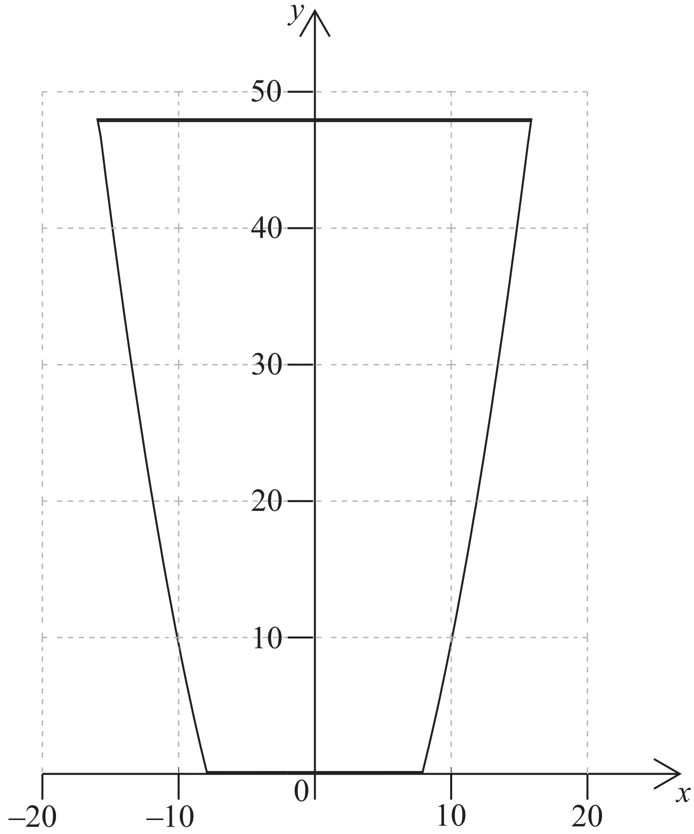
(c) Ava wins in one of her first three turns; *[4]*

(d) Ava eventually wins. *[4]*

*Do **NOT** write solutions on this page.*

13. *[Maximum mark: 16]*

The vertical cross-section of a container is shown in the following diagram.



The curved sides of the cross-section are given by the equation $y = 0.25x^2 - 16$. The horizontal cross-sections are circular. The depth of the container is 48 cm.

(a) If the container is filled with water to a depth of h cm, show that the volume, V cm^3, of the water is given by $V = 4\pi\left(\frac{h^2}{2} + 16h\right)$. *[3]*

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(Question 13 continued)

The container, initially full of water, begins leaking from a small hole at a rate given by $\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{250\sqrt{h}}{\pi(h+16)}$ where t is measured in seconds.

(b) (i) Show that $\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{250\sqrt{h}}{4\pi^2(h+16)^2}$.

(ii) State $\frac{\mathrm{d}t}{\mathrm{d}h}$ and hence show that $t = \frac{-4\pi^2}{250}\int\left(h^{\frac{3}{2}} + 32h^{\frac{1}{2}} + 256h^{-\frac{1}{2}}\right)\mathrm{d}h$.

(iii) Find, correct to the nearest minute, the time taken for the container to become empty. (60 seconds = 1 minute) *[10]*

Once empty, water is pumped back into the container at a rate of $8.5\,\mathrm{cm^3\,s^{-1}}$. At the same time, water continues leaking from the container at a rate of $\frac{250\sqrt{h}}{\pi(h+16)}\mathrm{cm^3\,s^{-1}}$.

(c) Using an appropriate sketch graph, determine the depth at which the water ultimately stabilizes in the container. *[3]*

Turn over

*Do **NOT** write solutions on this page.*

14. *[Maximum mark: 11]*

In triangle ABC,

$$3\sin B + 4\cos C = 6 \text{ and}$$
$$4\sin C + 3\cos B = 1.$$

(a) Show that $\sin(B+C) = \frac{1}{2}$. *[6]*

Robert conjectures that $\text{C}\hat{\text{A}}\text{B}$ can have two possible values.

(b) Show that Robert's conjecture is incorrect by proving that $\text{C}\hat{\text{A}}\text{B}$ has only one possible value. *[5]*